Kuwait University | Math 101 | Date: January 13, 2007
Dept. of Math. & Comp. Sci. | Final Exam | Duration: Two Hours

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Calculators, mobile phones, pagers and all other mobile communication equipment are not allowed

Answer the following questions. Each question weighs 4 points.

1. Evaluate the following limit, if it exists: $\displaystyle\lim_{x\to 0} \frac{\tan(2x) + x^3 \sin\left(\frac{4}{x}\right)}{x}$.

2. Let

$$f(x) = \begin{cases} \dfrac{6x-6}{|x-1|} & , \text{if } x < 1 \\[2ex] \dfrac{x^3-1}{\sqrt{x}-1} & , \text{if } x > 1. \end{cases}$$

   (a) Show that f is discontinuous at $x = 1$. (b) Classify this discontinuity as removable, jump or infinite.

3. Let

$$f(x) = \begin{cases} x^2 & , \text{if } x \le 1 \\[1ex] (2-x)^3 & , \text{if } x > 1. \end{cases}$$

   Find the local maxima and the local minima of f.

4. Let $f(x) = \frac{3}{8}(8 - x^2)\, x^{\frac{2}{3}}$. Find the x-coordinate of the point at which the tangent line to the graph of f is horizontal and the x-coordinate of the point at which the tangent line to the graph of f is vertical.

5. Evaluate:

   (a) $\displaystyle\int \frac{x+1}{\sqrt[3]{3x^2+6x+5}}\, dx.$ (b) $\displaystyle\int_0^{\frac{\pi}{2}} \sin^6 x \, \cos x \, dx.$

6. The graph of $y = f(x)$ intersects the line $y = x$ at $x = 0$ and $x = 1$. Find $f(x)$, if $f''(x) = 1 + 2x - 3x^2$.

7. Find an equation of the tangent line to the curve $y = 7 + \displaystyle\int_3^{4x-x^2} \frac{t}{t^2+1}\, dt$ at $x = 1$.

8. Find the average value, f_{av}, of $f(x) = 1 + \sqrt{4 - x^2}$ on $[-2, 2]$.

9. Find the area of the region bounded by the curves $x = y^2$ and $x = -2y^2 + 3$.

10. The region bounded by the curves $y = x^2$ and $y = 4$ is revolved about:

   (a) the line $y = -1$,

   (b) the line $x = 5$.

   Set up an integral that can be used to find the volume of the resulting solid in each case.

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Dept. of Math. & Comp. Sci. — Final Exam — Answers Key

1. For $x \neq 0$, $-1 \leq \sin\frac{4}{x} \leq 1$ and $-x^2 \leq x^2 \sin\frac{4}{x} \leq x^2$. Since, $\lim\limits_{x\to 0} x^2 = 0 = \lim\limits_{x\to 0}(-x^2)$, then from the Squeeze Theorem $\lim\limits_{x\to 0} x^2 \sin\frac{4}{x} = 0$.

$$\lim_{x\to 0}\frac{\tan(2x) + x^3\sin\left(\frac{4}{x}\right)}{x} = \lim_{x\to 0}\frac{\tan(2x)}{x} + \lim_{x\to 0}\frac{x^3\sin\frac{4}{x}}{x} = 2\lim_{x\to 0}\frac{\tan(2x)}{2x} + \lim_{x\to 0} x^2\sin\frac{4}{x} =$$
$$2\times 1 + 0 = \boxed{2}$$

2. (a) f is undefined at $x = 1$, (also, $\lim\limits_{x\to 1} f(x)$ does not exist), so it is discontinuous.

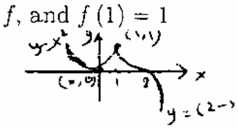
(b) $\lim\limits_{x\to 1^+} f(x) = \lim\limits_{x\to 1^+}\frac{(x-1)(x^2+x+1)}{\sqrt{x}-1}\times\frac{\sqrt{x}+1}{\sqrt{x}+1} = 6$ & $\lim\limits_{x\to 1^-} f(x) = -6$. f has a *jump* discontinuity at $x = 1$.

3. For $x > 1$, $f'(x) = -3(2-x)^2$, and for $x < 1$, $f'(x) = 2x$. The critical numbers of f are $\boxed{0, 1, 2}$.

I	$(-\infty, 0)$	$(0, 1)$	$(1, 2)$	$(2, \infty)$
sign of $f'(x)$	$-$	$+$	$-$	$-$
Conclusion	↘	↗	↘	↘

. $f(0) = 0$ is a local minimum of f, and $f(1) = 1$ is a local maximum of f.

Another solution: from the graph of f, $f(0) = 0$ is a local minimum of f, and $f(1) = 1$ is a local maximum of f.



4. $f'(x) = \frac{2 - x^2}{x^{\frac{1}{3}}}$. For horizontal tangent: $f'(x) = 0 \implies \boxed{x = \pm\sqrt{2}}$.

For vertical tangent: f' is undefined $\implies x = 0$ (f is continuous at $x = 0$).

5. (a) Put $u = 3x^2 + 6x + 5 \implies du = (6x + 6)\,dx$

$$\int\frac{x+1}{\sqrt[3]{3x^2+6x+5}}\,dx = \frac{1}{6}\int\frac{du}{\sqrt[3]{u}} = \frac{1}{4}u^{\frac{2}{3}} + C = \boxed{\frac{1}{4}(3x^2+6x+5)^{\frac{2}{3}} + C}$$

(b) Put $u = \sin x \implies u(0) = 0$, $u\left(\frac{\pi}{2}\right) = 1$ & $du = \cos x\,dx$

$$\int_0^{\frac{\pi}{2}}\sin^6 x\,\cos x\,dx = \int_0^1 u^6\,du = \left[\frac{u^7}{7}\right]_0^1 = \boxed{\frac{1}{7}}$$

6. The points of intersection are: $(0,0)$ and $(1,1)$ *.i.e.*, $f(0) = 0$ and $f(1) = 1$.

$f'(x) = \int f''(x)\,dx = \int\left(1 + 2x - 3x^2\right)dx = x + x^2 - x^3 + C_1$.

$f(x) = \int f'(x)\,dx = \int\left(x + x^2 - x^3 + C_1\right)dx = \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + C_1\,x + C_2$. Since, $f(0) = 0$, then $C_2 = 0$. Since, $f(1) = 1$, then $C_1 = \frac{5}{12}$.

Thus, $\boxed{f(x) = \frac{5x}{12} + \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4}}$

7. $y' = \dfrac{4x - x^2}{(4x - x^2)^2 + 1}\,(4 - 2x) \implies y'|_{x=1} = \dfrac{3}{5}$ & $y|_{x=1} = 7 + \displaystyle\int_3^3 \frac{t}{t^2+1}\,dt = 7$. Equation of the tangent: $\boxed{y - 7 = \frac{3}{5}(x - 1) \text{ or } 5y = 3x + 32}$

8. $\displaystyle\int_a^b f(x)\,dx = \int_{-2}^{2} \left(1 + \sqrt{4 - x}\right)\,dx = \int_{-2}^{2} dx + \int_{-2}^{2} \sqrt{4 - x}\,dx = x|_{-2}^{2} + \tfrac{1}{2}\pi(2)^2 = \boxed{4 + 2\pi}$

$f_{av} = \frac{1}{b-a}\displaystyle\int_a^b f(x)\,dx = \frac{1}{2-(-2)}\int_{-2}^{2}\left(1 + \sqrt{4 - x}\right)\,dx = \boxed{\dfrac{\pi + 2}{2}}$

9. The points of intersection are: $(1, 1)$ & $(1, -1)$. A is an R_y- region.

Area $= \displaystyle\int_{-1}^{1} [(-2y^2 + 3) - (y^2)\ ]\,dy = \int_{-1}^{1} (-3y^2 + 3)\,dy$

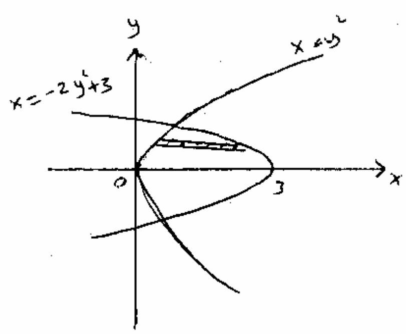
$= 2\displaystyle\int_0^1 (-3y^2 + 3)\,dy = 2\left[\frac{-3y^3}{3} + 3y\right]_0^1 = \boxed{4}$

10. (a) Revolution about the line $y = -1$:

Washer's Method: $Volme = \pi \displaystyle\int_{-2}^{2} \left[(5)^2 - (x^2 + 1)^2\right]\,dx$

(b) Revolution about the line $x = 5$:

(II) Cylindrical shell's Method: $Volme = 2\pi \displaystyle\int_{-2}^{2} (5 - x)\,(4 - x^2)\,dx.$



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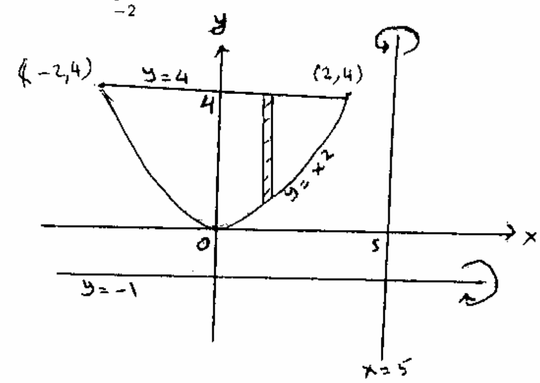

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